Quantum walk based applications

Combinatorial Optimization with Hybrid Quantum-Classical Algorithms

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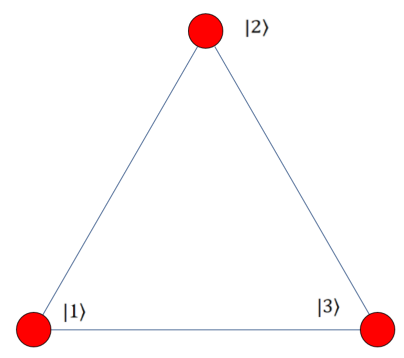
Introduction

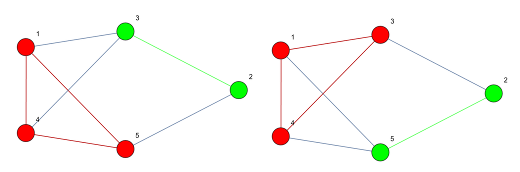
Quantum walk on a graph.

Example K_3, a triangle, nodes $V = 1,2,3$, links $E = \{(1,2),(1,3),(2,3)\}$ (undirected)



- classical random walk:
  - at the beginning walker at a node $v \in V$
  - repeat the following:
    if the walker is at node v, choose randomly a neighbour of v and move to it
- quantum walk (QW):
  - basis states are vertices $|v_i\rangle$
  - define a permutation π of vertices, such that $\pi : v_i \to v_j$ indicates that $v_i = v_j$ or (v_i, v_j) is a link
  - define a unitary transformation corresponding to π: $S_\pi := \sum_{i \in V} |\pi(i)\rangle \langle i|$
  - make a collection of such permutations $\mathcal{P}$ such that all possible links (in both directions) are covered at least by one permutation (for any $(v,w) \in E$ there is a $\pi \in \mathcal{P}$ s.t. $\pi(v) = w$)
  - take a coin state for each π, $|\pi\rangle$
  - take a coin unitary operator C (e.g. 'Grover's coin': $|s\rangle = \frac{1}{\sqrt{n}} \sum_\pi |\pi\rangle$, $C = 2|s\rangle\langle s| - I$)
  - coined QW unitary: $U = \sum_{\pi \in \mathcal{P}} S_\pi |\pi\rangle \langle \pi| C$
  - quantum walk: select a initial state of walker and coin $|\psi_0\rangle$, at time $t = 0,1,2,\cdots$ QW is in a state: $|\psi_t\rangle := U^t |\psi_0\rangle$
  - continuous time QW, A adjacency matrix of a graph (symmetric binary matrix), $U(t) := e^{-iAt}$, $t \in \mathbb{R}$ is unitary operator, continuous time QW: $|\psi_t\rangle = U(t)|\psi_0\rangle$
  - QW on directed graphs (A. Montanaro 2007): cover all nodes of graphs with directed cycles, s.t. every node is in exactly one cycle (self-loops are accepted), use corresponding permutation π for constructing S_π. Choose similarly many such permutations so that every directed link is covered at least by one permutation. If this is possible, we get a coined quantum walk
  - staggered QW: find a 'tessellation' $\mathcal{T}_i$ of the graph, grouping of all nodes in disjoint sets (partition) in such a way that within each set the corresponding subgraph is fully connected (see Fig. bellow), within each set create a Grower's diffusion operator add them up and use it as a H in e^{-itH} Find several tessellations which together cover all links of the graph and make product of corresponding unitaries with different parameters $t_i \in \mathbb{R}$ and use it as QW unitary evolution.

Two tessellations (shown with red and green colors), of a graph which cover all links of the graph



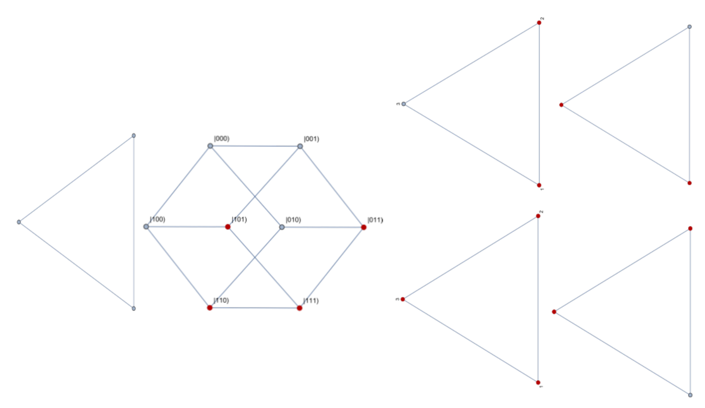
Possible applications of QW

We consider applications of QW related to optimization and network analysis [4]. QW is known to be on the background of such famous algorithms like Grover's search, which can be seen as a QW on complete graph and QAOA (Quantum Approximate Optimization Algorithm) which incorporates a continuous time QW on a hypercube.

Quantum walk assisted QAOA

The mixing phase of standard QAOA is quantum walk on hyper-cube in which nodes are bit-strings encoding a solution of the optimization problem. In some optimization problems one have to take into account that not all bit-strings are meaningful solutions. In this case one should replace QW on hyper-cube by some other QW which only travels in the solution subspace. [3]. Our preliminary work: A sample problem of finding minimal vertex cover of K_3-graph.

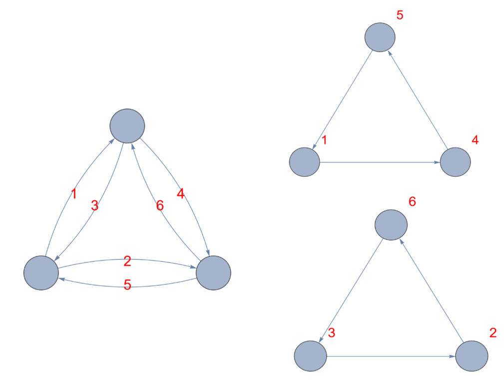
The figure: from left to right, K_3 graph, corresponding hyper-cube in which nodes indicate bit-strings as suggested solutions, red vertices are true vertex covers, 4 triangles show the possible vertex covers, 3 of which are minimal.



- continuous time QW on the solution space using a circulant graph spanned over solution space
  - has quantum circuit implementation using quantum Fourier transform
  - analytical test on K_3: perfect solution with one layer of QW-assisted QAOA
- use staggered QW on a graph spanned on solution space
  - a discrete time QW
  - such QW assisted QAOA solves K_3-problem perfectly in just one layer of QAOA

Non-backtracking quantum walk

Non-backtracking (NB) random walk (classical) and matrices related to it have been under intensive study in recent years. For instance, in our recent studies we found an interesting application of such topic to rigorous spectral clustering of sparse weighted graphs [2, 1]. NB random walk deviates from usual by forbidding reversing a step of a walk. NB walk can be described by so called non-backtracking matrix, which is a binary matrix (B) in which entries are between directed links in the graph and entry $B_{(ij),(jl)} = 1$, indicates that subsequent steps $i \to j \to k$ is allowed by NB walk, and otherwise the entry is 0. The B matrix is quadratic not necessarily symmetric. It can be seen as an adjacency matrix of an directed network. NB walk is equivalent to the walk on this directed network. We use this fact to define a quantum NB walk (QNB-walk) assuming that the corresponding conditions for QW on such a directed network are fulfilled.

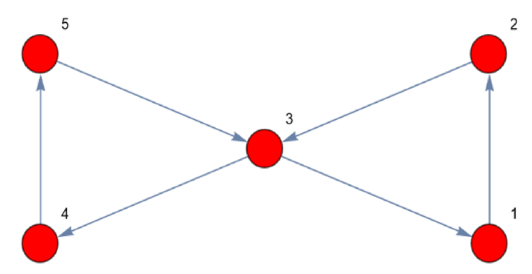


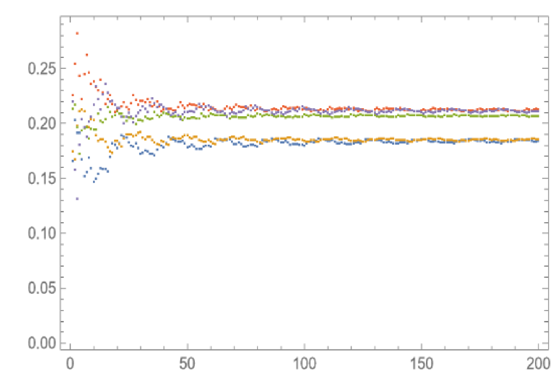
As an example we considered QNB-walk on K_3, triangle. Next picture shows how the corresponding directed graph is constructed from K_3 in which links are bidirectional (left-hand side). The NB-graph constitutes of two disjoint directed triangles. The corresponding QNB-walk can be solved exactly with analytical solution for the spectral decomposition of the QW unitary operator.

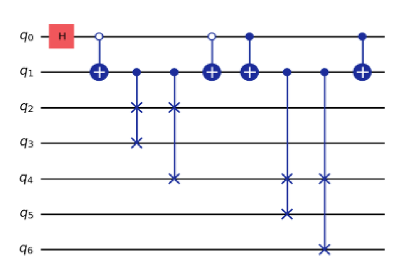
Quantum gate implementation of the QW on a directed graph

- any n-cycle can is equivalent to $n-1$ subsequent transpositions (2-cycles)
- a cycle $(a,b,c,\cdots,d,g)$ is a permutation: $\begin{pmatrix} a & b & c & \cdots & g \\ g & a & b & \cdots & d \end{pmatrix}$
- can be decomposed into transpositions $(a,b),(b,c),\cdots,(d,g)$
- each transposition (i,j) is like $(|i\rangle\langle j| + |j\rangle\langle i| + \sum_{k \neq i,j} |k\rangle\langle k|)$,
- each transposition is implemented as a SWAP gate
- coin is used to control which cycles are implied $\to$ we get controlled SWAP gates (Fredkin gates)

The figure shows first simulations on a 'tie-bow' graph. It has two directed cycles and five nodes. The plot shows time-average probability of finding a walker in a particular node up to 200 step. A quantum circuit for such QW is shown, $x - x$ is the swap gate. The qubit q_0 is the coin and q_1 Ancilla.







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Unitary Zero Space

Cumucore

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Summary

- **QAOA is a major algorithm for solving optimization problems on NISQ devices**
- **We suggested a new way of implementing QW assisted QAOA using staggered QW**
- **Defined a novel non-bactracking QW for network analysis**
- **Promising first simulations but needs more experimental/theoretical evidence**

References

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